ON THE 2ND PART OF HILBERT'S 5TH PROBLEM (OR, ON FUNCTIONAL EQUATIONS AND THE RELATED DIFFERENTIAL EQUATIONS): AN UP-TO-DATE VIEW FROM CAUCHY PROBLEMS (WITH A GENERALIZATION OF A RESULT OF A. KOLMOGOROV AND M. NAGUMO)

ROBERTO d'ANGIO

Institute for Quantitative Methods, Faculty of Engineering, Università C. Cattaneo, Castellanza (VA), Italy

Abstract: Given a functional equation and its solution, e.g., $(*)$ below (one of our four equations)

$$H(x+y) = H(x) + K(x,\, x+y), \quad H(x_0) = c_0 \quad (x,\, y,\, x_0 \in [0,\, \infty)) \tag{$*$}$$

$$H(x) = \int_{x_0}^{x} h(t)\,dt + c_0, \quad K(x,\, x+y) = \int_{x}^{x+y} h(t)\,dt \quad (x \in [x_0,\, \infty),\, x_0,\, y \in [0,\, \infty))$$

(viz., an equation, where the unknown functions *H* and *K* occur free from differential and integral operators) which is solved (via differentiation of the same) by the reduction to a differential equation, the second part of Hilbert's fifth problem [7] (HP#5.2, for short) requires the solution of *the same or suitably modified* functional equation with *much weaker regularity conditions* than the differentiability of the unknown functions (e.g. continuity, measurability, etc.) *in such a way* that the solution is *the same or similarly modified*. This talk presents the strongest ever solution of HP#5.2 in that two initial value problems for functional equations (solved by the reduction to ordinary and partial differential equation Cauchy problems) are given such that the suitably modified functional equation, e.g., the discrete analogue of $(*)$ above

$$H(n+m) = H(n) + K(n,\, n+m), \quad H(n_0) = c_0 \quad (n,\, m,\, n_0 \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}) \tag{$**$}$$

needs *no regularity condition at all* for its solution – that is the *weakest* ever *regularity condition* in the HP#5.2 sense. Further, the solutions with no regularity conditions are the *discrete analogue* of those in the differentiability case, see, e.g., $(*)$, since the Riemann-integral operator $\int$ and the continuous arbitrary function $h : [0,\, \infty) \to \mathbb{R}$ occurring in the latter are replaced by the summation operator Σ

and an arbitrary sequence $x : \mathbb{N} \to \mathbb{R}$ in the former. Initial value problems for functional equations have not occurred thus far in the functional equation literature, see the References. In particular, the functional equations referred to above are in two unknowns, and the partial differential equations arising from differentiating them are actually (linear first order) mixed partial-ordinary differential equations in two unknowns, too. Finally, as a by-product, the solutions of the said functional equations initial value problems and Cauchy problems generalise, with fewer assumptions or "axioms", the well-known Kolmogorov [9] and Nagumo [12] four-axiom discrete mean-value.

References

[1] J. Aczél, Lectures on Functional Equations and their Applications, Academic Press, NY, 1966.

[2] J. Aczél, The state of the second part of Hilbert's fifth problem, Bull. Amer. Math. Soc. 20 (1989), 153-163.

[3] J. Aczél and J. Dhombres, Functional Equations in Several Variables, Cambridge University Press, Cambridge, 1991.

[4] R. Courant, Differential and Integral Calculus, Vol. I, Wiley, NY, 1988 (2nd German edition, 1937).

[5] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, Singapore, 2002.

[6] G. H. Hardy, Divergent Series, Oxford, Clarendon Press, 1949.

[7] D. Hilbert, Mathematical problems, Bull. Amer. Math. Soc. 37 (2000), 407-436 (reprinted from Bull. Amer. Math. Soc., 8 July, 1902, pp. 437-479).

[8] A. Járai, Regularity Properties of Functional Equations in Several Variables, Springer, NY, 2005.

[9] A. N. Kolmogorov, Sur la notion de la moyenne, Atti R. A. Lincei 12(6) (1930), 388-391.

[10] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, Birkhäuser, Basel, 2009.

[11] D. S. Mitrinovitć, Analytic Inequalities, Springer, Berlin, 1970.

[12] M. Nagumo, Über eine Klasse der Mittelwerte, Japan. J. Math. 7(7) (1930), 71-79.

[13] W. Rudin, Principles of Mathematical Analysis, McGraw-Hill International, 1982.

[14] M. Toda, R. Kubo and N. Saito, Statistical Physics, Vol. I, Springer, Berlin, 1998.